

New path integral representation for Hubbard model: II. Spinless case

V.Kirchanov, V. Zharkov ¹

*Perm State Technical university, Komsomolcky Prospect, 29a, Perm,
614600, Russia*

*Natural Sciences Institute of Perm State university,
Genkel st.4, Perm,614990, Russia.*

Abstract

The Hubbard model is used to study an electronic system. In this paper we present the new path integral representation for Hubbard model. We have constructed the new supercoherent state for spinless electrons which appears from a set of eigenfunctions of atomic limit of strongly correlated systems. Exact calculation of nonlinear representation of a supergroup has been carried. This group defines the transformation of atomic base. The general formalism we elaborate for Hubbard model is the one widely used in the gauge field theory of the nonlinear representation of a superconformal group.

¹vita@psu.ru, Kirchanv@rambler.ru

1 Introduction

The Hubbard model was originally constructed to describe a metal-insulator transition for spin-dependent fermions in a simple way [1, 2, 3].

Today this model is still remain the main workspace for investigation of the strong electron's correlation. There exist many approaches to this model for describing many electrons system: the band limit approximation for the weak interaction between electrons and the atomic limit for electrons with the strong coulomb repulsion. We start the series of papers in which we are intending to elaborate the new approach to the Hubbard model.

We will present a new path integral approach to the strong interaction regime. The main ingredients for us is the usage of a supercoherent state acting upon an atomic base and the work with effective functional for an electronic system. We will develop the procedure of geometric quantization for strongly correlated systems of electrons. Let us make the brief sketch of the program firstly developed in the series of papers of one of the authors [4]. One of the main distinction of this approach to the Hubbard model is that we treat the local supergroup of the local space-time as the main object of our theory. This supergroup generates the transformation of the space-time coordinates which assigns the arguments for any function describing this system. This representation of a supergroup in the superspinor space has to contain a Lorents or an $SO(4)$ generators equal to the even subalgebra of the Hubbard operators. Odd Hubbard operators produce some superextension of the Lorents group into some supergroup. This dynamic supergroup is given by local dynamic superfields describing the local degrees of freedom of the strongly correlated electrons system. We shall introduce the supercoherent state depending on generalized angles equal to the bosonic and fermionic fields of the system. Parametrised by x,y,z,t space-time manifold which determines the arguments of wave function defines us the spinor and superspinor bundle. This superbundle is determined by the supercoherent state.

There are two kinds of the gauge fields in superspinor bundle: one sort of fields is the composite fields equal to the quadratic combination of odd grassmann fields and other sort are nonlinear fields determining the local coordinates frame of the four-dimensional space-time. In general, the local superspinor bundle defines the nonlinear representation of a superconformal group as a maximal group of 4-dimentional space-time of interacting fermionic system.

As the first step of quantization of electronic system with strong coulomb repulsion we perform the reformulation of Hubbard model into the atomic limit formalism. This approach is well known in Mott-Hubbard insulators theory. We want to point out that our approach includes all elements of geometric quantisation [5]: for example possess the algebra of 4-dimensional rotations (Lorentz) group, Cartan differential one-forms which give us the lagrangian of system, the nonlinear representation of underlying supergroup as a ground for the supercoherent state.

In the second paper of this series we will fulfill the exact calculation of the nonlinear representation of spinless supergroup generators of which appears in the so called "tower of symmetry" in the Hubbard model [4]. This gives us the possibility to introduce the supercoherent state for some deformed version of spinless algebra of the strongly correlated electron system and as a result to obtain the effective action in a future paper.

Let us give the brief description of the calculation method for the finding representation

of dynamic supergroup in strongly interacting models. Our task is to find in this model such group structure which could help us to describe the specificity of strong correlation. We take the following construction as a base:

1) we will collect space coordinates together with the time coordinate and will consider some curved space-time as a base in which Lorents subalgebra of superconformal group act in the spinor base on dynamical fields.

2) full bases of electronic operators gives us the supergroup which is parametrised by 2 dynamical fermionic fields: this fields comprise the real massless spinor.

3) the superspinor representation of a supergroup gives us the supercoherent state described by the nonlinear function over odd grassmanian fields. This function characterises the local properties of the strongly correlated system.

2 Atomic description of Hubbard model

We consider the Hubbard model:

$$H = -W \sum_{ij\sigma} \alpha_{\sigma,i}^+ \alpha_{\sigma,j} + U \sum_{i,\sigma} n_{\sigma,i} n_{-\sigma,i} + \mu \sum_{\sigma,i} n_{\sigma,i}, \quad (1)$$

here $\alpha_{\sigma,i}^+ \alpha_{\sigma,j}$ -electron creation and annihilation operators. $n_{\sigma,i}$ - electron density operator $W, U,$ -band width, one- site electron repulsion and chemical potential.

At first we represent the Hubbard model in atomic bases which determine the atomic limit. This limit appear as a result of the following procedure. A zero approximation of the atomic limit is described by one-site repulsion term:

$$U \sum_{i,\sigma} n_{\sigma,i} n_{-\sigma,i} + \mu \sum_{\sigma,i} n_{\sigma,i}.$$

This hamiltonian can be diagonalized by the following one-site atomic eigenfunction:

$$|0 \succ; |+\succ = \alpha_{\uparrow}^+ |0 \succ; |-\succ = \alpha_{\downarrow}^+ |0 \succ; |2 \succ = \alpha_{\uparrow}^+ \alpha_{\downarrow}^+ |0 \succ. \quad (2)$$

This bases gives us the fundamental representation of some supergroup in the space of dimension $(2, 2)$. Point out that the states $|+\succ; |-\succ;$ are fermions but $|0 \succ; |2 \succ$ are bosons in our construction. We insert later odd grassmann fields and make this fact obvious (ie states $|+\succ; |-\succ;$ will be depend on odd order of the grassmann fields but states $|0 \succ; |2 \succ$ -on even order of the grassmann fields). All operators in this bases will be the matrix which are determined by commutation and anticommutation relation giving some superalgebra. Full set of the Hubbard operators have 16 operators part of which

$$(X^{0+}, X^{0-}, X^{+0}, X^{-0}, X^{+2}, X^{-2}, X^{2+}, X^{2-})$$

are the fermionic operators, but other part

$$(X^{+-}, X^{-+}, X^{++} - X^{--}, X^{02}, X^{20}, X^{00} - X^{22})$$

—are the bosonic operators. X^{ij} -Hubbard operators contain only one non-zero element equal 1 sitting on site (i, j) in the matrix representation. Point out that this set of operators gives some bases for some superalgebra.

We have the following representation for creation-annihilating operators in this bases:

$$\alpha_{\uparrow}^+ = X^{+0} + X^{2-}\alpha_{\downarrow}^+ = X^{-0} + X^{2+} \quad (3)$$

The Hubbard model in this representation has the form:

$$H = U \sum_{i,p} X_i^{pp} - W \sum_{ij\alpha\beta} X_i^{-\alpha} X_j^{\beta} \quad (4)$$

3 Supercoherent state for Hubbard model

In constructing the supercoherent state we use the following interesting observation in interpretation of the set of atomic operators and function for on-site Hubbard repulsion. This observation can be formulated as the following statement: six even Hubbard operators constitute the subalgebra isomorphic with algebra of Lorentz group or algebra of four dimensional rotation group in spinor representation. Complete derivation of this statement will be obtain in subsequent paper.

To characterise the state of the system by coherent state we input some fields which depend on coordinates x, y, z and the time t . We have three component dynamic vector of the electric field

$$\mathbf{E} = (E^+(x, y, z, t), E^-(x, y, z, t), E^z(x, y, z, t)),$$

three component dynamical vector of the magnetic field

$$\mathbf{h} = (h^+(x, y, z, t), h^-(x, y, z, t), h^z(x, y, z, t))$$

and four component dynamical odd grassmann fields

$$\chi^*(x, y, z, t), \chi(x, y, z, t),$$

which are the fermionic fields giving the components of maiorana spinor. All dynamical fields appear in supercoherent state in the following manner:

$$|G\rangle = \exp \left[\begin{array}{cccc} E_z & \chi & \chi & E^+ \\ \chi^* & h_z & h^+ & \chi \\ \chi^* & h^- & -h_z & \chi \\ E^- & \chi^* & \chi^* & -E_z \end{array} \right] |0\rangle, \quad (5)$$

Exponent here act in space of atomic eigenfunctions ($|0\rangle, |+\rangle, |-\rangle, |2\rangle$), Function $|0\rangle$ is highest weight vector of representation space of supergroup which is given by exponent.

4 Evolution operator for electronic system

The transition amplitude of the evolution operator of the quantum systems is given by the following expresion: $\langle Z_f | e^{-iH(t_f - t_i)} | Z_i \rangle$. We want to obtain the expression for the effective functional using the states $|Z\rangle$. Time evolution of the system is given by the following operator:

$$U(t, t_0) = T_{ord} \exp(-i \int_{t_0}^t H(\tau) d\tau);$$

if $t - t_0 = \delta t$ is small, ie $\delta t \ll 1$, then

$$U(t_0 + \delta t, t_0) = 1 - i \int_{t_0}^{t_0 + \delta t} H(\tau) d\tau.$$

It follows from this expression that the symbol for evolutionary operator has the following form:

$$U(Z, Z^* | t_0 + \delta t, t_0) = \exp(-i \int_{t_0}^{t_0 + \delta t} H(Z, Z^* | \tau) d\tau).$$

We divide time interval $[t_0, t]$ by the number N and obtain N small intervals for finding the expression for symbol $U(Z, Z^* | t, t_0)$. Consider the matrix elements of evolution operator $\exp(-iH(t_f - t_i))$ between the states $\langle Z_f |$ and $|Z_i\rangle$. Factorising operator $\exp(-iH(t_f - t_i))$ by inserting the identity operator $\int d\mu(Z) |Z\rangle \langle Z| = 1$ we obtain the following representation:

$$\langle Z_f | \exp(-iH(t_f - t_i)) | Z_i \rangle = \int \prod_{k=1}^N d\mu(Z_k) \langle Z_f | Z_N \rangle$$

$$\langle Z_N | e^{-i\epsilon H} | Z_{N-1} \rangle \dots \langle Z_{k-1} | e^{-i\epsilon H} | Z_k \rangle \dots \langle Z_1 | e^{-i\epsilon H} | Z_i \rangle,$$

here $\epsilon = \frac{t_f - t_i}{N}$. In first order of ϵ we can transform this formula and place the symbol of operator H in the exponent

$$\frac{\langle Z_{k+1} | e^{-i\epsilon H} | Z_k \rangle}{\langle Z_{k+1} | Z_k \rangle} = \frac{\langle Z_{k+1} | (1 - i\epsilon H) | Z_k \rangle}{\langle Z_{k+1} | Z_k \rangle} = e^{-i\epsilon \frac{\langle Z_{k+1} | H | Z_k \rangle}{\langle Z_{k+1} | Z_k \rangle}} + O(\epsilon^2). \quad (6)$$

As a result we obtain the representation

$$\langle Z_f | e^{-iH(t_f - t_i)} | Z_i \rangle = \lim_{N \rightarrow \infty} \int \prod_{k=1}^N d\mu(Z_k) \langle Z_{k+1} | Z_k \rangle e^{-i\epsilon \sum_{k=1}^N \frac{\langle Z_{k+1} | H | Z_k \rangle}{\langle Z_{k+1} | Z_k \rangle}}, \quad (7)$$

here $|Z_0\rangle = |Z_i\rangle$; $\langle Z_{N+1}| = \langle Z_f|$. Let define a variation of the following type $|Z\rangle$: $|\delta Z_{k+1}\rangle = |Z_{k+1}\rangle - |Z_k\rangle$. We have:

$$\begin{aligned} \langle Z_f | e^{-iH(t_f - t_i)} | Z_i \rangle &= \lim_{N \rightarrow \infty} \int \prod_{k=1}^N [d\mu(Z_k) \langle Z_k | Z_k \rangle] \langle Z_f | Z_N \rangle \\ &\exp\left(\sum_{k=1}^N \left(\ln\left(1 - \frac{\langle Z_k | \delta Z_k \rangle}{\langle Z_k | Z_k \rangle}\right) - i\epsilon \frac{\langle Z_k | H | Z_{k-1} \rangle}{\langle Z_k | Z_{k-1} \rangle}\right)\right). \end{aligned}$$

in linear-slice approximation. We take the following expression for the time derivative: Considering the first order in ϵ we can take the following expression for the time derivative:

$$\frac{d|Z\rangle}{dt} = \frac{|\delta Z\rangle}{\epsilon}.$$

In first order in ϵ , we obtain the final path integral representation of the evolutionary operator in coherent state formalizm

$$\begin{aligned} \langle Z_f | e^{-iH(t_f-t_i)} | Z_i \rangle &= \int_{|Z(t_i)\rangle=|Z_i\rangle}^{|Z(t_f)\rangle=|Z_f\rangle} D(Z, Z^*) e^{-iS[Z, Z^*]}; \\ S[Z, Z^*] &= \int_{t_i}^{t_f} dt \int_V dr \left(\frac{\langle Z(r, t) | i \frac{\partial}{\partial t} - H | Z(r, t) \rangle}{\langle Z(r, t) | Z(r, t) \rangle} \right. \\ &\quad \left. - i[Ln(\langle Z_f | Z(t_f) \rangle) - Ln(\langle Z_i | Z(t_i) \rangle)] \right). \end{aligned} \quad (8)$$

Measure of integration is given by the following expression:

$$D[Z, Z^*] = \prod_{t_i < t < t_f} \prod_r d\mu[Z(r, t)^*, Z(r, t)] \langle Z(r, t) | Z(r, t) \rangle.$$

This form of path intergal representation will be the starting point of our consideration.

5 Nonlinear representation of supergroup in Hubbard model

In construction of supercoherent state we have the followng supermatrix which we must compute analytically:

$$U = \exp \begin{pmatrix} E_z & \chi & \chi & E^+ \\ \chi^* & h_z & h^+ & \chi \\ \chi^* & h^- & -h_z & \chi \\ E^- & \chi^* & \chi^* & -E_z \end{pmatrix}$$

In this expression we have the fields of different statistics: for example, set of the fields

$$(\chi^*(t, x, y, z), \chi(t, x, y, z))$$

are the odd grassman valued function of space-time coordinates and describe the fermionic degree of freedom but fields

$$(E_z(t, x, y, z), E^+(t, x, y, z), E^-(t, x, y, z), h_z(t, x, y, z), h^+(t, x, y, z), h^-(t, x, y, z))$$

describe the electro-magnetic degree of freedom, equal to two three component vectors of the space-time coordinates and are bosonic. In this paper we concentrate in calculating exact representation of exponent of the supermatrix in the coherent state. We take the dynamic electrical, magnetic and grassmann fields which depend on coordinates of 4-dimensional space-time manifold on definite the coordinates and omit (t, x, y, z) coordinates in subsiquent formulas.

Our general strategy will be to expand the supermatrix to N-order in fields. Then we can isolate and collect certain series and get some recurrent formular for general term in infinite series. Using this formula we can sum all terms to anytical compact representation. Analytical representation of the supermatrix elements will be the final point of our work. As a starting point we have the following exponential expression for the representation of super extension of the Lorentz group in the spinor representation.

Expanding this exponent in series we can obtain first and second order in the parameters b and h . We see that the polynomial series on the grassmann numbers can be classified in grassmann order n . All the supermatrix elements can be represented as a coefficients in grassmann polynomials of order n , where $n=0,1,2,3,4$.

6 Matrix series for the nonlinear representation of supergroup

First of all point out that the supermatrix in exponent have two submatrix: one is the odd grassmann matrix and the other is the even submatrix containing only the fields of type $E_i(t, x, y, z)$ and $h_i(t, x, y, z)$ type. As a first step we make expansion of the exponent for the even matrix. Expanding in series this exponent we can obtain first, second order and 3,4,5 order in bosonic fields $E_i, i = 1, 2, 3$. and $h_i, i = 1, 2, 3$. For example, the series for $n=0, 1, 2$ has the following form:

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \\ & \begin{pmatrix} E_z & 0 & 0 & E^+ \\ 0 & h_z & h^+ & 0 \\ 0 & h^- & -h_z & 0 \\ E^- & 0 & 0 & -E_z \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} b^2 & 0 & 0 & 0 \\ 0 & h^2 & 0 & 0 \\ 0 & 0 & h^2 & 0 \\ 0 & 0 & 0 & b^2 \end{pmatrix} + \\ & \frac{1}{3!} \begin{pmatrix} b^2 E_z & 0 & 0 & b^2 E^+ \\ 0 & h^2 h_z & h^2 h^+ & 0 \\ 0 & h^2 h^- & -h^2 h_z & 0 \\ b^2 E^- & 0 & 0 & -b^2 E_z \end{pmatrix} + \frac{1}{4!} \begin{pmatrix} b^4 & 0 & 0 & 0 \\ 0 & h^4 & 0 & 0 \\ 0 & 0 & h^4 & 0 \\ 0 & 0 & 0 & b^4 \end{pmatrix} + \\ & \frac{1}{5!} \begin{pmatrix} b^4 E_z & 0 & 0 & b^4 E^+ \\ 0 & h^4 h_z & h^4 h^+ & 0 \\ 0 & h^4 h^- & -h^4 h_z & 0 \\ b^4 E^- & 0 & 0 & -b^4 E_z \end{pmatrix} \end{aligned}$$

here we introduce the following abbrivation $b = \sqrt{E_z^2 + E^+ E^-}, h = \sqrt{h_z^2 + h^+ h^-}$

For the odd grassmann number we have following expansion series of exponent. We write here two terms for the grassmann fields (χ^*, χ) for obtaining coefficients in higher order in E_i and h_i .

$$\begin{pmatrix} 0 & \chi & \chi & 0 \\ \chi^* & 0 & 0 & \chi \\ \chi^* & 0 & 0 & \chi \\ 0 & \chi^* & \chi^* & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} 0 & \chi^* E^+ + \chi p1 & \chi^* E^+ + \chi p3 & 0 \\ \chi^* E_z + \chi(E^- + h^+ + h_z) & 0 & 0 & \chi^* E^+ + \chi(-E_z + h^+ + h_z) \\ \chi^* E_z + \chi(E^- + h^- - h_z) & 0 & 0 & \chi^* E^+ + \chi(-E_z + h^- - h_z) \\ 0 & \chi^* p2 + \chi E^- & \chi^* p4 + \chi E^- & 0 \end{pmatrix}$$

For even order of the grassmann variables we have following series for the composite bosonic fields:

$$\begin{pmatrix} 2\chi\chi^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{3!} \begin{pmatrix} 4E_z\chi\chi^* + (h^+ + h^-)\chi\chi^* & 0 & 0 & 0 \\ 0 & -2E_z\chi\chi^* & -2E_z\chi\chi^* & 0 \\ 0 & -2E_z\chi\chi^* & -2E_z\chi\chi^* & 0 \\ 0 & 0 & 0 & -4E_z\chi\chi^* + (h^+ + h^-)\chi\chi^* \end{pmatrix} + \frac{1}{4!} \begin{pmatrix} 2mm\chi\chi^* & 0 & 0 & 4E^+E_z\chi\chi^* \\ 0 & -2E_z(2h_z + X)\chi\chi^* & -4E_zh^+\chi\chi^* & 0 \\ 0 & -4E_zh^-\chi\chi^* & -2E_z(-2h_z + X)\chi\chi^* & 0 \\ -4E^-E_z\chi\chi^8 & 0 & 0 & 2mm\chi\chi^* \end{pmatrix}$$

here $m = e^2 + h^2$, $X = h^+ + h^-$, $pm = h_z + h^-$, $pp = h_z + h^+$, $mp = h^+ - h_z$, $mm = h^- - h_z$ and $mm = m + 2E_z^2 + E_zX$ The expansion series for third order of the grassmann variable has the following form of first two terms

$$\frac{m}{4!} * \begin{pmatrix} 0 & E^+\chi^* + p1\chi & E^+\chi^* + p3\chi & 0 \\ E^-\chi + (E_z + pp)\chi^* & 0 & 0 & E^+\chi^* + (-E_z + pp)\chi \\ E^-\chi + (E_z + mm)\chi^* & 0 & 0 & E^+\chi^* + (-E_z + mm)\chi \\ 0 & E^-\chi + p2\chi^* & E^-\chi + p4\chi^* & 0 \end{pmatrix}$$

here $p1 = E_z + pm$, $p2 = -E_z + pm$, $p3 = E_z + mp$, $p4 = -E_z + mp$

Expanding this series on 12 order in even and odd parameters we can obtain the analytical representation for the supermatrix elements.

7 Analytical representation of supergroup

Having series representation for supergroup for high order in fields we can obtain the general analytical form for the matrix elements. Our task is to obtain exact dependence of the matrix element over component of fields: $E_z, E^+, E^-; h_z, h^+, h^-$ but not dependence over b and h .

For example the general form of u_{11} supermatrix elements in orders higher than 8 as a functions of the dynamic even and odd grassmanian fields are given by following expressions

$$u_{11} = \cosh(e) + E_z \sinh(e)/e + (2f_3 + Xf_2 + 2E_z(2f2EE + Xf1EE) + 2E_z^2(2f1EE + XfEE))\chi\chi^*$$

here the coefficients $f_i, i = 2, 3, 4$ are some series in e and h variables:
 $f2EE, f1EE, fEE; f2h$ have some infinite series over e and h variables.
 Another matrix element equal to

$$u_{21} = \chi^*(f_4 + (E_z + h^+ + h_z)f_3 + E_z(h^+ + h_z)f_2) + \chi E^-(f_3 + (h^+ + h_z)f_2)$$

and we have some function for the coefficients: f_4, f_3, f_2

For u_{31} we have

$$u_{31} = \chi^*(f_4 + (E_z + h^- - h_z)f_3 + E_z(h^- - h_z)f_2) + \chi E^-(f_3 + (h^- + h_z)f_2)$$

Last element of first column is

$$u_{41} = E^- \sinh(e)/e + 2E_z E^-(2f1EE + XfEE)\chi\chi^*$$

For coefficients in this expression we have: $X = h^+ + h^-$

For second column we have

$$u_{12} = \chi(f_4 + (E_z + h^- + h_z)f_3 + E_z(h^- + h_z)f_2) + \chi^* E^+(f_3 + (h^- + h_z)f_2)$$

For u_{22} we have:

$$u_{22} = \cosh(h) + h_z \sinh(h)/h + E_z(f2h + Xh_z fhh + (2h_z + X)f1hh)(-2\chi\chi^*)$$

For u_{32} we have:

$$u_{32} = h^- \sinh(h)/h + E_z(f_2 + 2h^- f1hh + h^- X fhh)(-2\chi\chi^*)$$

For u_{42} we have:

$$u_{42} = \chi^*(f_4 + (-E_z + h^- + h_z)f_3 - E_z(h^- + h_z)f_2) + \chi E^-(f_3 + (h^- + h_z)f_2)$$

For u_{13} we have:

$$u_{13} = \chi(f_4 + (E_z + h^+ - h_z)f_3 + E_z(h^+ - h_z)f_2) + \chi^* E^+(f_3 + (h^+ - h_z)f_2)$$

For u_{23} we have:

$$u_{23} = h^+ \sinh(h)/h + E_z(f_2 + 2h^+ f_1 h h + h^+ X f h h)(-2\chi\chi^*)$$

For u_{33} we have:

$$u_{33} = \cosh(h) - h_z \sinh(h)/h + E_z(f_2 h - X h_z f h h + (-2h_z + X) f_1 h h)(-2\chi\chi^*)$$

For u_{43} we have:

$$u_{43} = \chi^*(f_4 + (-E_z + h^+ - h_z)f_3 - E_z(h^+ - h_z)f_2) + \chi E^-(f_3 + (h^+ - h_z)f_2)$$

For u_{14} we have:

$$u_{14} = E^+ \sinh(e)/e + 2E_z E^+(2f_1 E E + X f E E)\chi\chi^*$$

For u_{24} we have:

$$u_{24} = \chi(f_4 + (-E_z + h^+ + h_z)f_3 - E_z(h^+ + h_z)f_2) - \chi^* E^+(f_3 + (h^+ + h_z)f_2)$$

For u_{34} we have:

$$u_{34} = \chi(f_4 + (-E_z + h^- - h_z)f_3 - E_z(h^- - h_z)f_2) + \chi^* E^+(f_3 + (h^- - h_z)f_2)$$

For u_{44} we have:

$$u_{44} = \cosh(e) - E_z \sinh(e)/e + (2f_3 + X f_2 - 2E_z(2f_2 E E + X f_1 E E) + 2E_z^2(2f_1 E E + X f E E))\chi\chi^*$$

We see later that many series in our list are equivalent to each other. After such selection between similar ones we have only some different series.

8 Analytical representation for series

Collecting the terms in series expansion for a and b coefficients to 12 order we obtain for example the following representation for:

$$f_2 = \frac{1}{3!} + \frac{b^2 + h^2}{5!} + \frac{b^4 + h^2 b^2 + h^4}{7!} + \frac{b^6 + h^2 b^4 + h^4 b^2 + h^6}{9!} + \frac{b^8 + h^2 b^6 + h^4 b^4 + h^6 b^2 + h^8}{11!} + \frac{b^{10} + h^2 b^8 + h^4 b^6 + h^6 b^4 + h^8 b^2 + h^{10}}{13!} +$$

$$\frac{b^{12} + h^2 b^{10} + h^4 b^8 + h^6 b^6 + h^8 b^4 + h^{10} b^2 + h^{12}}{15!} +$$

$$\frac{b^{14} + h^2 b^{12} + h^4 b^{10} + h^6 b^8 + h^8 b^6 + h^{10} b^4 + h^{12} b^2 + h^{14}}{17!} + \dots$$

Let us show how to sum following infinite series for b for example. We have

$$1 + \frac{b^2}{2!} + \frac{b^4}{4!} + \frac{b^6}{6!} + \frac{b^8}{8!} + \frac{b^{10}}{10!} + \dots + E_z(1 + \frac{b^2}{3!} + \frac{b^4}{5!} + \frac{b^6}{7!} + \frac{b^8}{9!} + \frac{b^{10}}{11!} + \frac{b^{12}}{13!} + \dots)$$

It is seen that the first series equal to

$$\cosh(b) = 1 + \frac{b^2}{2!} + \frac{b^4}{4!} + \frac{b^6}{6!} + \frac{b^8}{8!} + \frac{b^{10}}{10!} + \dots$$

and the second series equal to

$$\sinh(b)/b = 1 + \frac{b^2}{3!} + \frac{b^4}{5!} + \frac{b^6}{7!} + \frac{b^8}{9!} + \frac{b^{10}}{11!} + \frac{b^{12}}{13!} + \dots$$

For sum of two series we have the following expression $\cosh(b) + E_z \sinh(b)/b$

The main series for us is the following expansion:

$$f = \frac{1}{5!} + \frac{b^2 + h^2}{7!} + \frac{b^4 + h^2 b^2 + h^4}{9!} + \frac{b^6 + h^2 b^4 + h^4 b^2 + h^6}{11!} + \frac{b^8 + h^2 b^6 + h^4 b^4 + h^6 b^2 + h^8}{13!} +$$

$$\frac{b^{10} + h^2 b^8 + h^4 b^6 + h^6 b^4 + h^8 b^2 + h^{10}}{15!} +$$

$$\frac{b^{12} + h^2 b^{10} + h^4 b^8 + h^6 b^6 + h^8 b^4 + h^{10} b^2 + h^{12}}{17!} +$$

$$\frac{b^{14} + h^2 b^{12} + h^4 b^{10} + h^6 b^8 + h^8 b^6 + h^{10} b^4 + h^{12} b^2 + h^{14}}{19!} + \dots$$

Let us show how the summation of this series can be performed. We can make the summation of subpart of hole series:

$$\frac{b^5}{5!} + \frac{b^7}{7!} + \frac{b^9}{9!} + \frac{b^{11}}{11!} + \frac{b^{13}}{13!} + \dots = \sinh(b) - b - \frac{b^3}{3!}$$

$$h^2(\frac{b^7}{7!} + \frac{b^9}{9!} + \frac{b^{11}}{11!} + \frac{b^{13}}{13!} + \dots) = h^2(\sinh(b) - b - \frac{b^3}{3!} - \frac{b^5}{5!})$$

$$h^4(\frac{b^9}{9!} + \frac{b^{11}}{11!} + \frac{b^{13}}{13!} + \dots) = h^4(\sinh(b) - b - \frac{b^3}{3!} - \frac{b^5}{5!} - \frac{b^7}{7!})$$

Having this series representation we can rewrite expression for f

$$\begin{aligned} & \frac{1}{5!} + \frac{b^2 + h^2}{7!} + \frac{b^4 + h^2b^2 + h^4}{9!} + \frac{b^6 + h^2b^4 + h^4b^2 + h^6}{11!} + \frac{b^8 + h^2b^6 + h^4b^4 + h^6b^2 + h^8}{13!} + \\ & \quad \frac{b^{10} + h^2b^8 + h^4b^6 + h^6b^4 + h^8b^2 + h^{10}}{15!} + \\ & \quad \frac{b^{12} + h^2b^{10} + h^4b^8 + h^6b^6 + h^8b^4 + h^{10}b^2 + h^{12}}{17!} + \dots = \end{aligned}$$

$$\frac{1}{b^5}(\sinh(b) - b - \frac{b^3}{3!}) + \frac{1}{b^7}h^2(\sinh(b) - b - \frac{b^3}{3!} - \frac{b^5}{5!}) + \frac{1}{b^9}h^4(\sinh(b) - b - \frac{b^3}{3!} - \frac{b^5}{5!} - \frac{b^7}{7!}) =$$

$$\begin{aligned} & \sinh(b)/b^5(1 + \frac{h^2}{b^2} + \frac{h^4}{b^4} + \dots) + (-b - \frac{b^3}{3!})/b^5(1 + \frac{h^2}{b^2} + \frac{h^4}{b^4} + \dots) + \\ & (-\frac{b^5}{5!})\frac{h^2}{b^7}(1 + \frac{h^2}{b^2} + \frac{h^4}{b^4} + \dots) + (-\frac{b^7}{7!})\frac{h^4}{b^9}(1 + \frac{h^2}{b^2} + \frac{h^4}{b^4} + \dots) + \dots \end{aligned}$$

It is seen that the series of the type $1 + \frac{h^2}{b^2} + \frac{h^4}{b^4} + \dots$ describe geometric series and gives the following result:

$$1 + \frac{h^2}{b^2} + \frac{h^4}{b^4} + \dots = \frac{1}{1 - \frac{h^2}{b^2}} = \frac{b^2}{b^2 - h^2}$$

If we insert this result we obtain the representation for:

$$\begin{aligned} & \frac{1}{5!} + \frac{b^2 + h^2}{7!} + \frac{b^4 + h^2b^2 + h^4}{9!} + \frac{b^6 + h^2b^4 + h^4b^2 + h^6}{11!} + \frac{b^8 + h^2b^6 + h^4b^4 + h^6b^2 + h^8}{13!} + \\ & \quad \frac{b^{10} + h^2b^8 + h^4b^6 + h^6b^4 + h^8b^2 + h^{10}}{15!} + \dots \end{aligned}$$

$$= \frac{\frac{\sinh(b)}{b^3}}{b^2 - h^2} + (-b - \frac{b^3}{3!})/b^5 \frac{b^2}{b^2 - h^2} + (-\frac{b^5}{5!})\frac{h^2}{b^7} \frac{b^2}{b^2 - h^2} + (-\frac{b^7}{7!})\frac{h^4}{b^9} \frac{b^2}{b^2 - h^2} + \dots =$$

$$\frac{\frac{\sinh(b)}{b^3}}{b^2 - h^2} - \frac{\frac{1}{b^2}}{b^2 - h^2} + (-\frac{1}{3!} - \frac{h^2}{5!} - \frac{h^4}{7!} - \dots)\frac{1}{b^2 - h^2} =$$

$$\frac{\frac{\sinh(b)}{b^3}}{b^2 - h^2} - \frac{\frac{1}{b^2}}{b^2 - h^2} + (-\sinh(h)/h^3 + \frac{1}{h^2})\frac{1}{b^2 - h^2} =$$

$$\frac{\frac{\sinh(b)}{b^3} - \frac{\sinh(h)}{h^3}}{b^2 - h^2} + \frac{1}{b^2h^2}$$

Let us consider two series: one is f and second is f_1 . If we multiply f by coefficient a^5 and make following substitution $b \rightarrow ba, h \rightarrow ha$ we obtain the following series

$$\begin{aligned}
& a^5 \frac{1}{5!} + a^7 \frac{b^2 + h^2}{7!} + a^9 \frac{b^4 + h^2 b^2 + h^4}{9!} + a^{11} \frac{b^6 + h^2 b^4 + h^4 b^2 + h^6}{11!} + \\
& a^{13} \frac{b^8 + h^2 b^6 + h^4 b^4 + h^6 b^2 + h^8}{13!} + \\
& a^{15} \frac{b^{10} + h^2 b^8 + h^4 b^6 + h^6 b^4 + h^8 b^2 + h^{10}}{15!} + \\
& a^{17} \frac{b^{12} + h^2 b^{10} + h^4 b^8 + h^6 b^6 + h^8 b^4 + h^{10} b^2 + h^{12}}{17!} + \dots
\end{aligned}$$

It is obvious that if we take derivative we can reduce factorial in our series for example:

$$f_1 = \frac{\partial(a^5 f)}{\partial a}, \quad f_2 = \frac{\partial f_1}{\partial a}, \quad f_3 = \frac{\partial f_2}{\partial a}, \quad f_4 = \frac{\partial f_3}{\partial a}$$

here we must insert $b \rightarrow ab, h \rightarrow ah$ and a put to 1 after differentiation.

Taking derivatives we obtain the analytical expression for f_i :

$$\begin{aligned}
f_1 &= \frac{\frac{\cosh(b)}{b^2} - \frac{\cosh(h)}{h^2}}{b^2 - h^2}; \quad f_2 = \frac{\frac{\sinh(b)}{b} - \frac{\sinh(h)}{h}}{b^2 - h^2}; \quad f_3 = \frac{\cosh(b) - \cosh(h)}{b^2 - h^2}; \\
f_4 &= \frac{b \sinh(b) - h \sinh(h)}{b^2 - h^2}
\end{aligned}$$

Series for f_i have the following forms:

$$\begin{aligned}
f_1 &= \frac{1}{4!} + \frac{b^2 + h^2}{6!} + \frac{b^4 + h^2 b^2 + h^4}{8!} + \frac{b^6 + h^2 b^4 + h^4 b^2 + h^6}{10!} + \frac{b^8 + h^2 b^6 + h^4 b^4 + h^6 b^2 + h^8}{12!} + \\
& \frac{b^{10} + h^2 b^8 + h^4 b^6 + h^6 b^4 + h^8 b^2 + h^{10}}{14!} + \\
& \frac{b^{12} + h^2 b^{10} + h^4 b^8 + h^6 b^6 + h^8 b^4 + h^{10} b^2 + h^{12}}{16!} + \\
& \frac{b^{14} + h^2 b^{12} + h^4 b^{10} + h^6 b^8 + h^8 b^6 + h^{10} b^4 + h^{12} b^2 + h^{14}}{17!} + \dots
\end{aligned}$$

$$\begin{aligned}
f_2 &= \frac{1}{3!} + \frac{b^2 + h^2}{5!} + \frac{b^4 + h^2 b^2 + h^4}{7!} + \frac{b^6 + h^2 b^4 + h^4 b^2 + h^6}{9!} + \frac{b^8 + h^2 b^6 + h^4 b^4 + h^6 b^2 + h^8}{11!} + \\
& \frac{b^{10} + h^2 b^8 + h^4 b^6 + h^6 b^4 + h^8 b^2 + h^{10}}{13!} + \\
& \frac{b^{12} + h^2 b^{10} + h^4 b^8 + h^6 b^6 + h^8 b^4 + h^{10} b^2 + h^{12}}{15!} + \\
& \frac{b^{14} + h^2 b^{12} + h^4 b^{10} + h^6 b^8 + h^8 b^6 + h^{10} b^4 + h^{12} b^2 + h^{14}}{17!} + \dots
\end{aligned}$$

$$\begin{aligned}
f_3 = & \frac{1}{2!} + \frac{b^2 + h^2}{4!} + \frac{b^4 + h^2b^2 + h^4}{6!} + \frac{b^6 + h^2b^4 + h^4b^2 + h^6}{8!} + \frac{b^8 + h^2b^6 + h^4b^4 + h^6b^2 + h^8}{10!} + \\
& \frac{b^{10} + h^2b^8 + h^4b^6 + h^6b^4 + h^8b^2 + h^{10}}{12!} + \\
& \frac{b^{12} + h^2b^{10} + h^4b^8 + h^6b^6 + h^8b^4 + h^{10}b^2 + h^{12}}{14!} + \\
& \frac{b^{14} + h^2b^{12} + h^4b^{10} + h^6b^8 + h^8b^6 + h^{10}b^4 + h^{12}b^2 + h^{14}}{16!} + \dots
\end{aligned}$$

$$\begin{aligned}
f_4 = & 1 + \frac{b^2 + h^2}{2!} + \frac{b^4 + h^2b^2 + h^4}{4!} + \frac{b^6 + h^2b^4 + h^4b^2 + h^6}{6!} + \frac{b^8 + h^2b^6 + h^4b^4 + h^6b^2 + h^8}{8!} + \\
& \frac{b^{10} + h^2b^8 + h^4b^6 + h^6b^4 + h^8b^2 + h^{10}}{10!} + \\
& \frac{b^{12} + h^2b^{10} + h^4b^8 + h^6b^6 + h^8b^4 + h^{10}b^2 + h^{12}}{12!} + \\
& \frac{b^{14} + h^2b^{12} + h^4b^{10} + h^6b^8 + h^8b^6 + h^{10}b^4 + h^{12}b^2 + h^{14}}{14!} + \dots
\end{aligned}$$

If we take the series for f and multiply it by b^2 and take following derivative $\frac{\partial(b^2f)}{\partial(b^2)}$ we obtain the series for fEE .

$$\begin{aligned}
fEE = & \frac{1}{5!} + \frac{2b^2 + h^2}{7!} + \frac{3b^4 + 2h^2b^2 + h^4}{9!} + \frac{4b^6 + 3h^2b^4 + 2h^4b^2 + h^6}{11!} + \\
& \frac{5b^8 + 4h^2b^6 + 3h^4b^4 + 2h^6b^2 + h^8}{13!} + \\
& \frac{6b^{10} + 5h^2b^8 + 4h^4b^6 + 3h^6b^4 + 2h^8b^2 + h^{10}}{15!}
\end{aligned}$$

For series fhh we must make multiplication of f on h^2 and make derivative on h^2 :
 $fhh = \frac{\partial(h^2f)}{\partial(h^2)}$

$$\begin{aligned}
fhh = & \frac{1}{5!} + \frac{2h^2 + b^2}{7!} + \frac{3h^4 + 2b^2h^2 + b^4}{9!} + \frac{4h^6 + 3b^2h^4 + 2b^4h^2 + b^6}{11!} + \\
& \frac{5h^8 + 4b^2h^6 + 3b^4h^4 + 2b^6h^2 + b^8}{13!} + \\
& \frac{6h^{10} + 5b^2h^8 + 4b^4h^6 + 3b^6h^4 + 2b^8h^2 + b^{10}}{15!}
\end{aligned}$$

$$f_2EE = \frac{1}{3!} + \frac{2b^2 + h^2}{5!} + \frac{3b^4 + 2h^2b^2 + h^4}{7!} + \frac{4b^6 + 3h^2b^4 + 2h^4b^2 + h^6}{9!} +$$

$$\frac{5b^8 + 4h^2b^6 + 3h^4b^4 + 2h^6b^2 + h^8}{11!} + \frac{6b^{10} + 5h^2b^8 + 4h^4b^6 + 3h^6b^4 + 2h^8b^2 + h^{10}}{13!}$$

To evaluate f_1EE, f_2EE, f_1hh we multiply fEE by a^5 and make the following substitution $b \rightarrow ba, h \rightarrow ha$. After calculation we fix $a = 1$.

It is seen that expression for f_1EE, f_2EE, f_1hh are the following:

$$f_1EE = \left(\frac{\partial fEE}{\partial a} \right)_{a=1}$$

$$f_1EE = \frac{1}{4!} + \frac{2b^2 + h^2}{6!} + \frac{3b^4 + 2h^2b^2 + h^4}{8!} + \frac{4b^6 + 3h^2b^4 + 2h^4b^2 + h^6}{10!} + \frac{5b^8 + 4h^2b^6 + 3h^4b^4 + 2h^6b^2 + h^8}{12!} + \frac{6b^{10} + 5h^2b^8 + 4h^4b^6 + 3h^6b^4 + 2h^8b^2 + h^{10}}{14!} + \dots$$

Repeating all operation we can obtain for $f_2EE = \left(\frac{\partial^2 fEE}{\partial^2 a} \right)_{a=1}$

$$f_2EE = \frac{1}{3!} + \frac{2b^2 + h^2}{5!} + \frac{3b^4 + 2h^2b^2 + h^4}{7!} + \frac{4b^6 + 3h^2b^4 + 2h^4b^2 + h^6}{9!} + \frac{5b^8 + 4h^2b^6 + 3h^4b^4 + 2h^6b^2 + h^8}{11!} + \frac{6b^{10} + 5h^2b^8 + 4h^4b^6 + 3h^6b^4 + 2h^8b^2 + h^{10}}{13!}$$

and for $f_1hh = \left(\frac{\partial fhh}{\partial a} \right)_{a=1}$ and for $f_2hh = \left(\frac{\partial f_1hh}{\partial a} \right)_{a=1}$

$$f_1hh = \frac{1}{4!} + \frac{2h^2 + b^2}{6!} + \frac{3h^4 + 2b^2h^2 + b^4}{8!} + \frac{4h^6 + 3b^2h^4 + 2b^4h^2 + b^6}{10!} + \frac{5h^8 + 4b^2h^6 + 3b^4h^4 + 2b^6h^2 + b^8}{12!} + \frac{6h^{10} + 5b^2h^8 + 4b^4h^6 + 3b^6h^4 + 2b^8h^2 + b^{10}}{14!} + \dots$$

$$f_2hh = \frac{1}{3!} + \frac{3h^2 + b^2}{5!} + \frac{5h^4 + 3b^2h^2 + b^4}{7!} + \frac{7h^6 + 5b^2h^4 + 3b^4h^2 + b^6}{9!} + \frac{9h^8 + 7b^2h^6 + 5b^4h^4 + 3b^6h^2 + b^8}{11!} + \frac{11h^{10} + 9b^2h^8 + 7b^4h^6 + 5b^6h^4 + 3b^8h^2 + b^{10}}{13!} + \dots$$

In analytical form we have for $f_2h = 2f_2hh - f_2$

Series for $DEDEF$ equal to derivative of fEE on b^2 . It is seen if we compare series for fEE and $DEDEF$: $DEDEF = \frac{\partial fEE}{\partial (b^2)}$

$$DEDEF = \frac{2}{7!} + \frac{6b^2 + 2h^2}{9!} + \frac{12b^4 + 6h^2b^2 + 2h^4}{11!} + \frac{20b^6 + 12h^2b^4 + 6h^4b^2 + 2h^6}{13!} + \frac{30b^8 + 20h^2b^6 + 12h^4b^4 + 6h^6b^2 + 2h^8}{15!} + \dots$$

We can obtain the following representation for $DEDEF1$: $DEDEF1 = \frac{\partial f h h}{\partial(b^2)}$

$$DEDEF1 = \frac{2}{6!} + \frac{6b^2 + 2h^2}{8!} + \frac{12b^4 + 6h^2b^2 + 2h^4}{10!} + \frac{20b^6 + 12h^2b^4 + 6h^4b^2 + 2h^6}{12!} + \frac{30b^8 + 20h^2b^6 + 12h^4b^4 + 6h^6b^2 + 2h^8}{14!} + \dots$$

Comparing series for $DEDhf2$ and series for f_2 we can obtain $DEDhf2 = \frac{\partial^2 f_2}{\partial(b^2)\partial(h^2)}$

$$DEDhf2 = \frac{1}{7!} + \frac{2b^2 + 2h^2}{9!} + \frac{3b^4 + 4h^2b^2 + 3h^4}{11!} + \frac{4b^6 + 6h^2b^4 + 6h^4b^2 + 4h^6}{13!} + \frac{5b^8 + 8h^2b^6 + 9h^4b^4 + 8h^6b^2 + 5h^8}{15!} + \frac{6b^{10} + 10h^2b^8 + 12h^4b^6 + 12h^6b^4 + 10h^8b^2 + 6h^{10}}{17!} + \dots$$

and for $DEDhf3$ we can obtain the following representation: $DEDhf3 = \frac{\partial^2 f_3}{\partial(b^2)\partial(h^2)}$

$$DEDhf3 = \frac{1}{6!} + \frac{2b^2 + 2h^2}{8!} + \frac{3b^4 + 4h^2b^2 + 3h^4}{10!} + \frac{4b^6 + 6h^2b^4 + 6h^4b^2 + 4h^6}{12!} + \frac{5b^8 + 8h^2b^6 + 9h^4b^4 + 8h^6b^2 + 5h^8}{14!} + \frac{6b^{10} + 10h^2b^8 + 12h^4b^6 + 12h^6b^4 + 10h^8b^2 + 6h^{10}}{16!} + \dots$$

Summing our calculation let's write all analytical formula for the function in the matrix elements:

$$\begin{aligned} f_{EE} &= \frac{2 \sinh(h)b^3 + h(b^2 - h^2) \cosh(b)b + (h^3 - 3b^2h) \sinh(b)}{2b^3h(b^2 - h^2)^2}; \\ f_{1EE} &= \frac{-2b \cosh(b) + 2b \cosh(h) + (b^2 - h^2) \sinh(b)}{2b(b^2 - h^2)^2}; \\ f_{2EE} &= \frac{b(b^2 - h^2) \cosh(b) - (b^2 + h^2) \sinh(b) + 2bh \sinh(h)}{2b(b^2 - h^2)^2}; \\ f_2 &= \frac{h \sinh(b) - b \sinh(h)}{b^3h - bh^3}; \quad f_3 = \frac{\cosh(b) - \cosh(h)}{b^2 - h^2}; \\ f_4 &= \frac{b \sinh(b) - h \sinh(h)}{b^2 - h^2}; \quad f = \frac{\frac{\sinh(b)}{b^3} - \frac{\sinh(h)}{h^3}}{b^2 - h^2} \end{aligned}$$

9 Conclusion

We have calculated the exact representation of the supergroup as well as the supercoherent state in the Hubbard model. These constructions naturally appear in the strongly correlated electronic systems in the case of introducing atomic bases in limit of large on-site Hubbard repulsion. The dynamical supergroup which operates in a local superbundle determined by any on-site eigenfunction gives us the wave function in the form of a superspinor. This superspinor describes a local supercoordinates frame in the curved supermanifold. The operator spinor part acting in tangent and cotangent bundles of this supermanifold in supergroup can be reformulated in the terms of the atomic Hubbard operators. Next step lies in the calculation of effective functional of Hubbard model.

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